0. Screw theory

0.1. Rigid body movements (in 3D)

Def.: Let $p, q \in \mathbb{R}^3$ and a map $g: \mathbb{R}^3 \to \mathbb{R}^3$

Then g is a rigid body transform, if

(1) $\| g(q) - g(p) \| = \| q - p \|$ preserves distance

(2) $g_*(v \times w) = g_*(v) \times g_*(w)$ pres. orientation

where $g_*(v) = g_*(p) - g_*(q)$, $v = p - q$

$\Rightarrow \cancel{g_*} \, v_1^T \cdot v_2 \} = g_*(v)^T \cdot g_*(v_2)$ pres. angles

0.2. Rotations

Let $R_{ab} = \left[\vec{x_{ab}}, \vec{y_{ab}}, \vec{z_{ab}} \right]$

If we have right handed coordinate systems, then $\det(R) = 1$.

Def: $SO(3) = \{ R \mid RR^T = I, \det R = +1 \}$

is the special orthagonal group.

Def: The Configuration space of a system is a set Q such that $\forall q \in Q$ defines a valid configuration.

Bsp: $SO(3)$ is a configuration space for rotations in $\mathbb{R}^3$ with respect to a fixed frame.

The connection with mappings g:

$R_{ab} \in SO(3) \to$ define a mapping $g_{ab}: \mathbb{R}^3 \to \mathbb{R}^3$

Let q_b a point specifed w.r. to frame O_b, then

$g_{ab}(q_b) = R_{ab} \, q_b = q_a$, where q_a are the coordinates in the base frame O_a

Lemma: Rotations are rigid body transforms

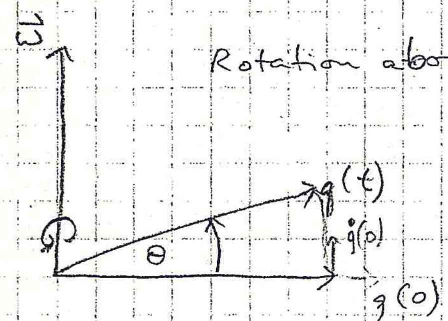
Proof: Exercise

Notation: $b \to a \times b$ is linear and can be written as $\hat{a} \cdot b$ where

$$\hat{a} = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}$$

0.2. Exponential coordinates for Rotations

Rotation about θ around $\vec{\omega}$



goal: express R as function of $\theta, \vec{\omega}$

Motivation: assume rotation with constant unit velocity, then

$$\dot{q}(t) = \vec{\omega} \times q(t) = \underline{\hat{\omega} \cdot q(t)}$$

Now integrate to obtain (because we have a linear diff. equation):

$$q(t) = e^{\hat{\omega} t} q(0)$$

where $e^{\hat{\omega} t} = I + \hat{\omega} t + \frac{(\hat{\omega} t)^2}{2!} + \frac{(\hat{\omega} t)^3}{3!} + \dots$ is matrix

$$\Rightarrow R(\vec{\omega}, \theta) = e^{\hat{\omega}\theta}$$, es gilt: $\boxed{\hat{\omega} = -\hat{\omega}^T}$

Def: $so(3) = \{ S \in \mathbb{R}^{3\times 3} \mid S^T = -S \}$

Then $so(3)$ is a vector space over $\mathbb{R}$ and every vector in $\mathbb{R}^3$ has a corresponding matrix in $so(3)$

Question: is the map $\exp :: so(3) \to SO(3)$ surjective?

i.e. given $R \in SO(3)$, do we find $\vec{\omega} \in \mathbb{R}^3$, $\|\omega\| = 1$, $\theta \in \mathbb{R}$ such that $R = \exp^{\hat{\omega}\theta}$?

Answer: yes!

Proof: given:

$$(*) \qquad R = \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{pmatrix}$$

Notation: $\cos\theta = c_\theta$

$\sin\theta = s_\theta$

$1 - \cos\theta = v_\theta$

$$(**) \qquad e^{\hat{\omega}\theta} = I + \hat{\omega}\sin\theta + \hat{\omega}^2(1-\cos\theta)$$

$$= \begin{pmatrix} \omega_1^2 v_\theta + c_\theta & \omega_1\omega_2 v_\theta - \omega_3 s_\theta & \omega_1\omega_3 v_\theta + \omega_2 s_\theta \\ \omega_1\omega_2 v_\theta + \omega_3 s_\theta & \omega_2^2 v_\theta + c_\theta & \omega_2\omega_3 v_\theta - \omega_1 s_\theta \\ \omega_1\omega_3 v_\theta - \omega_2 s_\theta & \omega_2\omega_3 v_\theta + \omega_1 s_\theta & \omega_3^2 v_\theta + c_\theta \end{pmatrix}$$

Equating $(*)$ and $(**)$ we solve for $\vec{\omega}$ and θ:

Because $\det R = 1 = \prod \lambda_i(R)$, λ_i eigenvalue of R and eigenvalues occur in complex conjugate pairs

$\Rightarrow \quad \|\lambda_i\| = 1$

$\Rightarrow \quad -1 \leq \operatorname{trace}(R) = \sum \lambda_i = r_{11} + r_{22} + r_{33} \leq 3$

$(***) \quad \Rightarrow \quad \theta = \cos^{-1}\left(\frac{\operatorname{trace}(R) - 1}{2}\right) \in [-1; 1]$

Note: Also $\theta \pm 2\pi n$ or $-\theta \pm 2\pi n$ can be chosen.

Now we get from the off-diagonal terms:

$$\left.\begin{aligned} r_{32} - r_{23} &= 2\omega_1 s_\theta \\ r_{13} - r_{31} &= 2\omega_2 s_\theta \\ r_{21} - r_{12} &= 2\omega_3 s_\theta \end{aligned}\right\} \overset{\theta \neq 0}{\Longrightarrow} \quad \vec{\omega} = \begin{pmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{pmatrix} \cdot \frac{1}{2 s_\theta}$$

Remark: For the choice $-\theta + 2\pi$ in (***) we obtain $-\vec{\omega}$ instead (flip of axis + backward rotation), ie there are 2 $(\vec{\omega}, \theta)$, $\theta \in [0, 2\pi)$ to generate R.

For $\theta = 0$, we have $R = I$ and $\vec{\omega}$ is arbitrary because $e^{\hat{\omega}\theta} = \begin{pmatrix} 1 & \\ & 1 \\ & & 1 \end{pmatrix}$ for any $\vec{\omega}$!

This constitutes a singularity because the map $R \to \vec{\omega}, \theta$ is not continuous at $R = I$!

Def: The components $\vec{\omega}, \theta$ are called exponential coordinates of R and canonical coordinates of $so(3)$.

Theorem: (Euler): Any orientation $R \in SO(3)$ is equivalent to a rotation about a fixed axis $\vec{\omega} \in \mathbb{R}^3$ through an angle $\theta \in [0, 2\pi)$.

0.4. Other (local) coordinates

Euler Angles ZYZ

To generate an arbitrary rotation R_{ab} between frames O_a, O_b do:

1) rotate by α about $z_b = z_a$
2) rotate by β about the new y_b
3) rotate by γ about the new z_b

$$\Rightarrow R_{ab} = R_{z,\alpha} R_{y,\beta} R_{z,\gamma}$$

$$= \begin{pmatrix} c_\alpha c_\beta c_\gamma - s_\alpha s_\gamma & -c_\alpha c_\beta s_\gamma - s_\alpha c_\gamma & c_\alpha s_\beta \\ s_\alpha c_\beta c_\gamma + c_\alpha s_\gamma & -s_\alpha c_\beta s_\gamma + c_\alpha c_\gamma & s_\alpha s_\beta \\ -s_\beta c_\gamma & s_\beta s_\gamma & c_\beta \end{pmatrix}$$

Questions: • is $(\alpha, \beta, \gamma) \longrightarrow R$ surjective? yes

• where are the singularities?

$\Rightarrow$ at $\beta = 0$, $\alpha = -\gamma$ $\Rightarrow R = I$

Euler ZYX angles

1) rotate by ϕ about $x_a = x_b$ (roll)
2) " by θ about old y_a (pitch)
3) " by ψ about old z_a (yaw)

Questions as before

Answer: Übung

Remark: There are <u>always</u> singularities in 3-dim. parametrisations of $SO(3)$.

0.5. Quaternions

Example of a global parametrisation of $SO(3)$ using 4 numbers:

A quaternion is a 4-vector $Q = (\underset{\in \mathbb{R}}{q_0}, \underset{\in \mathbb{R}^3}{\vec{q}})$ with

$$Q = q_0 + q_1 i + q_2 j + q_3 k \quad , \quad q_i \in \mathbb{R}$$

Properties:

$$i^2 = j^2 = k^2 = i \cdot j \cdot k = -1$$

$$ij = -ji = k \; ; \; jk = -kj = i \; ; \; ki = -ik = j$$

the conjugate is $Q^* = (q_0, -\vec{q})$

$\Rightarrow QQ^* = \|Q\|^2 = \Sigma q_i^2$

$\Rightarrow$ inverse $\dfrac{Q^*}{\|Q\|^2}$

Multiplikation:

$$Q \cdot P = (q_0 p_0 - \vec{q} \circ \vec{p} \, , \; q_0 \vec{p} + p_0 \vec{q} + \vec{q} \times \vec{p})$$

Quaternions form a vector space and group w.r. to multiplication.

Given a rotation $R_{ab} = R(\vec{\omega}, \theta)$ define

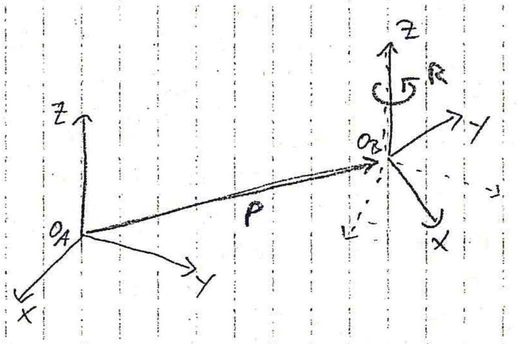
$$Q_{ab} = (\cos(\theta/2) \, , \; \vec{\omega} \sin(\theta/2))$$

Now $R_{ab} R_{bc} = R_{ac}$ can be identified with $Q_{ab} \cdot Q_{bc} = Q_{ac}$, ie the group structure of the quaternions make them an <u>efficient</u> representation of the $SO(3)$-group.

We can also inspect from a unit quaternion a rotation by

$$\Longrightarrow \quad Q = (q_0, \vec{q}) \longrightarrow \theta = 2\cos^{-1} q_0 \, , \quad \vec{\omega} = \begin{cases} \dfrac{\vec{q}}{\sin(\frac{\theta}{2})} & , \; \theta \neq 0 \\ 0 & \text{else} \end{cases}$$

0.6. General Rigid Motion



The general motion is composed of a translation and a rotation.

<u>Def.</u>: The configuration space $SE(3) = \{(p, R), p \in \mathbb{R}^3, R \in SO($
is called <u>special Euclidian group</u>.

As before we define a map $g_{ab}: \mathbb{R}^3 \to \mathbb{R}^3$ for every (p_{ab}, R_{ab}) to transform a frame O_A into O_B and coordinates backward:

$$g_{ab}(q_b) = p_{ab} + R_{ab}\, q_b = q_a$$

The corresponding map for vectors is:

$$g_*(v) = g(s) - g(r) = R_{ab}(s - r) = R\vec{v} \quad , \quad \vec{v} = s - r$$

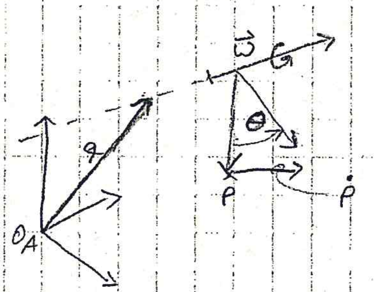
<u>Homogeneous Representation</u> (as 4x4 Matrix)

$\bar{q} = \begin{pmatrix} q \\ 1 \end{pmatrix}$, $T = \begin{bmatrix} R_{ab} & p_{ab} \\ 0 & 1 \end{bmatrix}$, $\bar{g}_{ab}(\bar{q}_b) = T\bar{q}$ <u>for points q.</u>

$\bar{v} = \begin{pmatrix} \vec{v} \\ 0 \end{pmatrix}$, $\bar{g}_*(v) = T\bar{v}$ <u>for vectors v.</u>

Remark: Perspective transforms and scaling by homogeneous matrices are <u>not</u> rigid body transforms and may <u>not</u> appear in $SE(3)$.

Exponential Coordinates for SE(3)



We assume that p rotates with unit velocity, then $\dot{p}(t) = \vec{\omega} \times (p(t) - q)$

Define $\hat{\xi} = \begin{bmatrix} \hat{\omega} & v \\ 0 & 0 \end{bmatrix}$, where $v = -\vec{\omega} \times q$ to obtain

$$\begin{bmatrix} \dot{p} \\ 0 \end{bmatrix} = \begin{bmatrix} \hat{\omega} & -\vec{\omega} \times q \\ 0 & 0 \end{bmatrix} \begin{bmatrix} p \\ 1 \end{bmatrix} = \hat{\xi}\, \bar{p}$$

and integration yields:

$$\underline{\bar{p}_a(t) = e^{\hat{\xi} t}\, \bar{p}_a(0)}$$

For pure translation with velocity v we define

$$\hat{\xi} = \begin{bmatrix} 0_{3\times3} & v \\ 0 & 0 \end{bmatrix} \text{ and get } \bar{p}(t) = e^{\hat{\xi} t}\, \bar{p}(0)$$

Def: An element of the set $se(3) = \{(v, \hat{\omega}),\ v \in \mathbb{R}^3, \hat{\omega} \in so(3)\}$ is called a <u>twist</u> and has homogeneous coordinates $\hat{\xi} = \begin{bmatrix} \hat{\omega} & v \\ 0 & 0 \end{bmatrix}$.

$\xi = (v, \vec{\omega})$ are the <u>twist coordinates</u> for $se(3)$.

<u>Proposition</u>: For given $\hat{\xi} \in se(3)$, the exponential map $\exp(\hat{\xi}\theta)$ yields an element of $SE(3)$.

<u>Proof</u>: 1) $\vec{\omega} = 0 \Rightarrow \hat{\xi}^2 = \hat{\xi}^3 = \hat{\xi}^i = 0$

$$\Rightarrow \exp(\hat{\xi}\theta) = I_{4\times4} + \hat{\xi}\theta = \begin{bmatrix} I_{3\times3} & v\theta \\ 0 & 1 \end{bmatrix} \quad \checkmark$$

(pure translation)

2) $\vec{\omega} \neq 0$, w.r. $\|\vec{\omega}\| = 1$

Define $g = \begin{bmatrix} I & \vec{\omega}\times v \\ 0 & 1 \end{bmatrix}$ and $\hat{\xi}' = g^{-1}\hat{\xi}g$

$$\hat{\xi}' = \begin{bmatrix} I & -\vec{\omega}\times v \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{\omega} & v \\ 0 & 0 \end{bmatrix} \begin{bmatrix} I & \vec{\omega}\times v \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \hat{\omega} & (\vec{\omega}\vec{\omega}^T)v \\ 0 & 0 \end{bmatrix}$$

As $\exp(\hat{\xi}\theta) = g\exp(\hat{\xi}'\theta)g^{-1}$ (→ <u>Übung</u>)

and $(\hat{\xi}')^2 = \begin{bmatrix} \hat{\omega}^2 & 0 \\ 0 & 0 \end{bmatrix}$

$(\hat{\xi}')^3 = \begin{bmatrix} \hat{\omega}^3 & 0 \\ 0 & 0 \end{bmatrix} \quad \ldots \quad (\hat{\xi}')^i = \begin{bmatrix} \hat{\omega}^i & 0 \\ 0 & 0 \end{bmatrix}$

$$\Rightarrow \exp(\hat{\xi}'\theta) = \begin{bmatrix} \overset{\text{rotation}}{\exp(\hat{\omega}\theta)} & (\vec{\omega}\vec{\omega}^T)v\theta \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow \exp(\hat{\xi}\theta) = \begin{bmatrix} \exp(\hat{\omega}\theta) & (I - \exp(\hat{\omega}\theta))\vec{\omega}\times v + \vec{\omega}\vec{\omega}^T v\theta \\ 0 & 1 \end{bmatrix} \in S$$

$$= g\exp(\hat{\xi}'\theta)g^{-1} \qquad \blacksquare$$

<u>Summary</u>: $p(\theta) = \exp(\hat{\xi}\theta)\,p(0)$ expresses the relative rigid movement ~~with respect~~ of a point $p(0)$ w.r. to the transform $\exp(\hat{\xi}\theta)$ and a single base frame.

<u>Question</u>: is $\exp : se(3) \to SE(3)$ surjective, i.e. can every element of $SE(3)$ be represented by a twist $\hat{\xi} \in se(3)$.

$\Longrightarrow$

<u>Answer:</u> yes!

<u>Proof:</u> 1) no rotation: $R = I$, set $\hat{\xi} = \begin{bmatrix} 0 & p/\|p\| \\ 0 & 0 \end{bmatrix}$, $\theta = \|p\|$

2) $R \neq I$:

- first solve $\exp(\hat{\omega}\theta) = R$ as before.
- Then solve $(I - \exp(\hat{\omega}\theta))\,\vec{\omega} \times v + \vec{\omega}\vec{\omega}^T v\theta = p$ for v

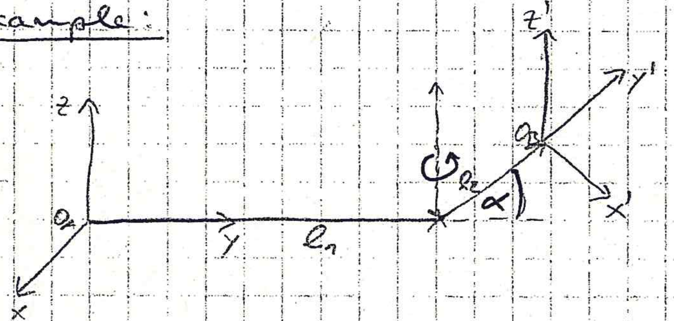
It can be shown that $\overbrace{(I - \exp(\hat{\omega}\theta))\hat{\omega} + \vec{\omega}\vec{\omega}^T\theta}^{M}$ can be inverted always

$\Rightarrow v = M^{-1} p$ ✓.

<u>Def.:</u> The vector $\xi\theta = \begin{bmatrix} v \\ \vec{\omega} \end{bmatrix}\theta \in \mathbb{R}^6$ is called exponential coordinates of the rigid transform $g \in SE(3)$.

<u>Remark:</u> The map is <u>not</u> unique!

<u>Example:</u>



Find twist coordinates for g_{ab} describing the rotation about a line.

$$g_{ab} = \begin{pmatrix} c_\alpha & -s_\alpha & 0 & -l_2 s_\alpha \\ s_\alpha & c_\alpha & 0 & l_1 + l_2 c_\alpha \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = (p_{ab}, R_{ab})$$

Assume $\alpha \neq 0$, $\vec{\omega} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, $\theta = \alpha \Rightarrow \exp(\hat{\omega}\theta) = R$

$$\hat{\omega} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Find translational part by

$$\left[(I - \exp(\hat{\omega}\theta))\hat{\omega} + \vec{\omega}\vec{\omega}^T\theta\right] v = p_{ab}$$

$\Rightarrow$

$$= \begin{bmatrix} -l\, s_\alpha \\ l_1 + l_2\, c_\alpha \\ 0 \end{bmatrix}$$

NR

$$\vec{\omega}\vec{\omega}^T = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$I - e^{\hat{\omega}\theta} = \begin{pmatrix} 1-c_\alpha & s_\alpha & 0 \\ -s_\alpha & 1-c_\alpha & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\Leftrightarrow \begin{pmatrix} s_\alpha & c_\alpha - 1 & 0 \\ 1-c_\alpha & s_\alpha & 0 \\ 0 & 0 & \alpha \end{pmatrix} v = p_{ab}$$

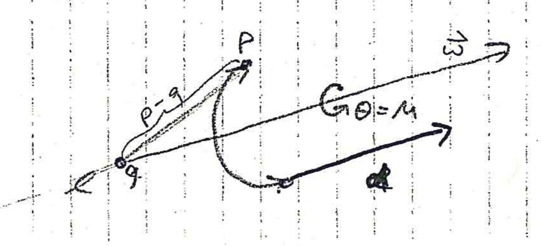
$$\Rightarrow v = \left(\frac{l_1 - l_2}{2} \,,\; \frac{(l_1 + l_2) s_\alpha}{2(1-c_\alpha)} \,,\; 0 \right)^T$$

$$\Rightarrow \xi = \begin{pmatrix} v \\ 0 \\ 0 \\ 1 \end{pmatrix} \alpha$$

<u>Remark:</u> p_{ab} is the translation of the origin of O_B with respect to the origin of O_A,
v appearing in the twist is <u>not</u> ~~the same~~ directly interpretable as translation!

0.7. Screws

<u>Def:</u> A screw consists of an axis l, a pitch $h = \frac{d}{\theta}$ and a ~~q~~ magnitude M.
It represents a <u>rotation</u> by an amount $\theta = M$ about l followed by <u>a translation</u> by an amount $h\theta = d$ parallel to l.



If $h = \infty$, i.e. $\theta = 0$, then the screw is a pure translation by a dist. M

We associate a rigid body transform g with the screw as

$$g(p) = q + \underline{\exp(\hat{\omega}\theta)(p-q)} + \underline{h\theta\vec{\omega}}$$

or in homogeneous coordinates:

$$\begin{bmatrix} \exp(\hat{\omega}\theta) & (I - \exp(\hat{\omega}\theta))q + h\theta\vec{\omega} \\ 0 & 1 \end{bmatrix} \begin{matrix} \in \mathbb{R}^{4\times 4} \\ \in SE(3) \end{matrix}$$

Now choose $v = -\vec{\omega} \times q + h\vec{\omega}$, then the twist $\xi = \binom{v}{\vec{\omega}}$ generates the screw motion. ($\|\omega\| = 1$, $\theta \neq 0$).

<u>Vise versa:</u> let $\hat{\xi} \in se(3)$ a twist with coordinates $(v, \vec{\omega})^T = \xi^T$ the define <u>screw coordinates</u> as:

1) Pitch: $\begin{cases} h = \dfrac{\vec{\omega}^T v}{\|\omega\|^2} & \text{falls } \vec{\omega} \neq 0 \\ h = \infty & \text{falls } \omega = 0 \end{cases}$

2) Axis: $l = \begin{cases} \dfrac{\vec{\omega} \times v}{\|\omega\|^2} + \lambda\vec{\omega}, \ \lambda \in \mathbb{R}, & \text{falls } \vec{\omega} \neq 0 \\ 0 + \lambda\vec{v} & \text{falls } \vec{\omega} = 0 \end{cases}$

3) Magnitude

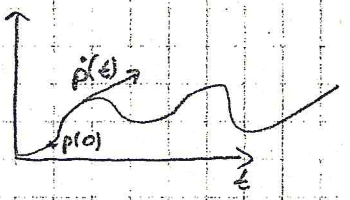
$$M = \begin{cases} \|\omega\|, & \text{if } \vec{\omega} \neq 0 \\ \|v\| & \text{if } \vec{\omega} = 0 \end{cases}$$

<u>Theorem</u>: Every rigid body motion in $SE(3)$ can be realised by a screw motion.

<u>Remark</u>:
- zero pitch screws are pure rotations and can model revolute joints.
- infinite pitch screws are pure translations and can model prismatic joints.
- soon: screws also model wrenches

0.8. Velocities

For point particles velocities are intuitiv:



velocities are the tangent vectors to the trajectory.

<u>Problem:</u> That does not generalize to rigid bodies!
Intuitive clear is pure translational or rotational velocity, but the mixture is more complicated and once more expressed as twists.

<u>Remark:</u> $\dot R_{ab} \notin so(3)$!!

Pure Rotation

$$q_a(t) = R_{ab}(t)\, q_b$$

$$v_{q_a}(t) = \underbrace{\dot R_{ab}(t)}_{\notin so(3)}\, q_b$$

$$= \underbrace{\dot R_{ab}(t)\, R_{ab}^{-1}(t)}_{\in so(3)}\, \underbrace{R_{ab}(t) q_b}_{q_a}$$

$$\Longrightarrow v_{q_a}(t) = \left(\dot R_{ab}(t)\, R_{ab}^{-1}(t)\right) \cdot q_a(t)$$

<u>Def:</u> $\hat\omega_{ab}^s = \dot R_{ab} R_{ab}^{-1}$ is called <u>instantaneous angular spatial velocity</u>, i.e. the velocity seen and expressed to the base frame O_A.

<u>Def:</u> $\hat\omega_{ab}^b = R_{ab}^{-1} \dot R_{ab}$ is called <u>instantaneous angular body velocity</u>.

<u>Lemma:</u> $\hat\omega_{ab}^b = R_{ab}^{-1}\, \hat\omega_{ab}^s\, R_{ab}$

$$v_{q_a} = \hat{\omega}^s_{ab} R_{ab} q_b = \vec{\omega}^s_{ab} \times q_a(t)$$

$$v_{q_b} = R^T_{ab} v_{q_a}(t) = \vec{\omega}^b_{ab} \times q_b(t)$$

General Case

Parametrise the motion as $g_{ab}(t) \in SE(3)$

i.e. $g_{ab} = \begin{bmatrix} R_{ab}(t) & p_{ab}(t) \\ 0 & 1 \end{bmatrix}$.

As before $\dot{g}_{ab}(t)$ is not useful, but $g_{ab}^{-1}\dot{g}_{ab}$ and $\dot{g}_{ab} g_{ab}^{-1}$ can be interpreted as velocities:

$$\underline{\dot{g}_{ab} g_{ab}^{-1}} = \begin{bmatrix} \dot{R}_{ab} & \dot{p}_{ab} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} R^T_{ab} & -R^T_{ab} p_{ab} \\ 0 & 1 \end{bmatrix} \in se(3)$$

<u>is a twist!</u>

<u>Def:</u> $\hat{V}^s_{ab} = \dot{g}_{ab} g_{ab}^{-1}$ is called the <u>spatial velocity</u> (with respect to the base (frame)) and has twist coordinates

$$V^s_{ab} = \begin{bmatrix} v^s_{ab} \\ \vec{\omega}^s_{ab} \end{bmatrix} = \begin{bmatrix} -\dot{R}_{ab} R^T_{ab} p_{ab} + \dot{p}_{ab} \\ (\dot{R}_{ab} R^T_{ab})^\vee \end{bmatrix} \rightarrow \vec{\omega} = (\hat{\omega})^\vee$$

$\downarrow$ Tex

<u>Def:</u> $\hat{V}^b_{ab} = g_{ab}^{-1} \dot{g}_{ab}$ is the body velocity, $V^b_{ab} = \begin{bmatrix} v^b_{ab} \\ \vec{\omega}^b_{ab} \end{bmatrix} = \begin{bmatrix} R^T \dot{p} \\ (R^T \dot{R})^\vee \end{bmatrix}$

We get: $V_{q_a} = \hat{V}^s_{ab} q_a = \vec{\omega}^s_{ab} \times q_a + v^s_{ab}$

<u>Remark:</u> For pure rotation or translation V^s_{ab} complies to the intuitive understanding, i.e.

$$V^s_{ab} = \begin{bmatrix} \dot{p}_{ab} \\ 0 \end{bmatrix} \quad \text{for pure translation}$$

$$V^s_{ab} = \begin{bmatrix} 0 \\ \vec{\omega}^s_{ab} \end{bmatrix} \quad \text{for pure rotation}$$

Zusatz:

Umrechnung $V^s_{ab} \longrightarrow V^b_{ab}$

$$\omega^s = R_{ab}\,\omega^b$$

$$\begin{aligned} v^s_{ab} &= p \times \omega^s_{ab} + \dot p \\ &= -\omega^s \times p + \dot p \\ &= -\hat\omega^s \cdot p + \dot p \\ &= -(\dot R R^T)^\vee \times p + \dot p \end{aligned}$$

$$\longrightarrow \quad \begin{aligned} &= p \times (R_{ab}\,\omega^b) + \dot p \\ &= p \times (R_{ab}\,\omega^b) + R\, v^b_{ab} \end{aligned}$$

$$\begin{bmatrix} v^s \\ \omega^s \end{bmatrix} = \underbrace{\begin{bmatrix} R & \hat p \cdot R \\ 0 & R \end{bmatrix}}_{Ad} \begin{bmatrix} v^b \\ v^s \end{bmatrix}$$

$$Ad_g^{-1} = Ad_{g^{-1}} \qquad \text{(s. Übung)}$$

$$\| $$

$$\begin{pmatrix} R^{-1} & -(\widehat{R^T p})R \\ 0 & R^{-1} \end{pmatrix} = \begin{pmatrix} R^T & -R^T \hat p \\ 0 & R^T \end{pmatrix}$$

$$\Rightarrow Ad^T_{g^{-1}_{oc_i}} = \begin{pmatrix} R & 0 \\ -\hat p^T R & R \end{pmatrix} = \begin{pmatrix} R & 0 \\ \hat p R & R \end{pmatrix}$$

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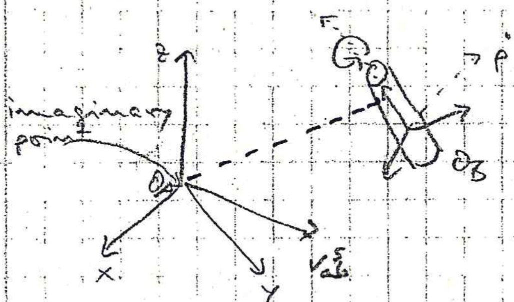
im 2D:

$$Ad_g = \begin{pmatrix} R & \begin{bmatrix} -p_x \\ p_y \end{bmatrix} \\ 0 & 1 \end{pmatrix} \qquad \begin{bmatrix} -p_x \\ p_y \end{bmatrix} \longrightarrow \bar p$$

$$\Rightarrow Ad^T_{g^{-1}} = \begin{bmatrix} R & 0 \\ -[\bar p]^T R & 1 \end{bmatrix}$$

The linear component v_{ab}^s is a mixture of translational and rotational parts.
v_{ab}^s has a geometrical meaning; it is the velocity of a point (possibly imaginary) attached to the body frame O_B traveling through the origin of O_A.



Lemma: • $\hat{V}_{ab}^s = \dot{g}_{ab} g^{-1} = g(g^{-1}\dot{g})g^{-1}$

$$= g_{ab}\,\hat{V}_{ab}^b\,g_{ab}^{-1}$$

• $$\vec{V}_{ab}^s = \begin{bmatrix} v_{ab}^s \\ \vec{\omega}_{ab}^s \end{bmatrix}$$

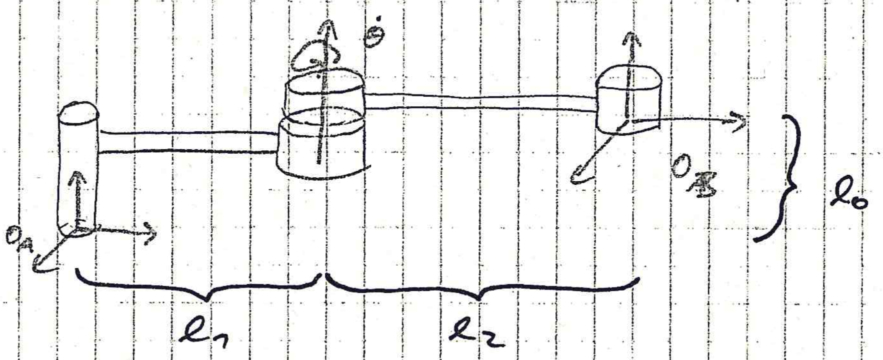
$$= \begin{bmatrix} R_{ab} & \hat{p}_{ab} R_{ab} \\ 0 & R_{ab} \end{bmatrix} \begin{bmatrix} v_{ab}^b \\ \vec{\omega}_{ab} \end{bmatrix}$$

$$= Ad(g_{ab})(V_{ab}^b)$$

$Ad\ g_{ab}$ maps twists with coordinates ξ from the frame O_A to O_B, if g_{ab} defines the configuration of O_B with respect to O_A.

Lemma: if $\hat{\xi} \in se(3)$ is ~~a~~ twist, then for any $g \in SE(3)$: $g\hat{\xi}g^{-1}$ is a twist with coordinates $Ad\ g\,\xi$.

Example:



$$g_{ab}(t) = \left(\begin{array}{ccc|c} c_\theta(t) & -s_\theta(t) & 0 & -l_2\, s_\theta(t) \\ s_\theta(t) & c_\theta(t) & 0 & l_1 + l_2 c_\theta(t) \\ 0 & 0 & 1 & l_0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right)$$

$$V^s = \begin{pmatrix} v^s \\ \vec\omega^s \end{pmatrix} = \begin{bmatrix} -\dot R R^T p + \dot p \\ (\dot R R^T)^\vee \end{bmatrix} = \begin{bmatrix} v^s \\ \begin{pmatrix} 0 \\ 0 \\ \dot\theta \end{pmatrix} \end{bmatrix}$$

$$v^s = -\dot\theta \begin{bmatrix} -s_\theta & -c_\theta & 0 \\ c_\theta & -s_\theta & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} c_\theta & s_\theta & 0 \\ -s_\theta & c_\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -l_2 s_\theta \\ l_1 + l_2 c_\theta \\ l_0 \end{bmatrix} + \begin{bmatrix} -l_2 c_\theta \\ -l_2 s_\theta \\ 0 \end{bmatrix} \dot\theta$$

$$= -\dot\theta \begin{bmatrix} -sc + sc & -s^2 - c^2 & 0 \\ c^2 + s^2 & sc - sc & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -l_2 s_\theta \\ l_1 + l_2 c_\theta \\ l_0 \end{bmatrix} - \dot\theta \begin{bmatrix} l_2 c_\theta \\ l_2 s_\theta \\ 0 \end{bmatrix}$$

$$= -\dot\theta \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} -l_2 s_\theta \\ l_1 + l_2 c_\theta \\ l_0 \end{bmatrix} - \begin{bmatrix} l_2 c_\theta \dot\theta \\ l_2 s_\theta \dot\theta \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -l_1 \dot\theta \\ 0 \\ 0 \end{bmatrix}$$

Rigid Body Movements: $g = (p, R) \in SE(3)$

↳ exponential coordinates $\xi\theta \in \mathbb{R}^6$ such that $e^{\hat{\xi}\theta} = g$ and $\xi = (v, \omega)$ twist coordinates,

$$\hat{\xi} = \begin{pmatrix} \hat{\omega} & v \\ 0 & 0 \end{pmatrix} \in \mathbb{R}^{4\times 4}$$

twist → screw (h, l, M)

pitch, axis, Magnitude

$$h = \frac{\omega^T v}{\|\omega\|^2},$$

$$l = \begin{cases} \frac{\omega \times v}{\|\omega\|^2} + \lambda \vec{\omega} & , \omega \neq 0 \\ 0 + \lambda v & , \omega = 0 \end{cases}$$

$$M = \begin{cases} \|\omega\| , & \omega \neq 0 \\ \|v\| , & \omega = 0 \end{cases}$$

screw → twist

pure rotation $\xi = \begin{pmatrix} -\omega \times q \\ \omega \end{pmatrix} \theta$

pure translation $\xi = \begin{pmatrix} v \\ 0 \end{pmatrix} \theta$

Velocities

spatial: $\hat{V}^s_{ab} = \dot{g}_{ab} g^{-1}_{ab} \Rightarrow \begin{pmatrix} v^s_{ab} \\ \omega^s_{ab} \end{pmatrix} = \begin{pmatrix} -\dot{R}R^T p + \dot{p} \\ (\dot{R}R^T)^\vee \end{pmatrix}$

body: $\hat{V}^b_{ab} = g^{-1}_{ab} \dot{g}_{ab}$

↳ $\hat{V}^s_{ab} = g_{ab} \hat{V}^b_{ab} g^{-1}_{ab}$

$\Rightarrow V^s_{ab} = \mathrm{Ad}_g V^b_{ab}, \quad \mathrm{Ad}_g = \begin{pmatrix} R & \hat{p}R \\ 0 & R \end{pmatrix}$

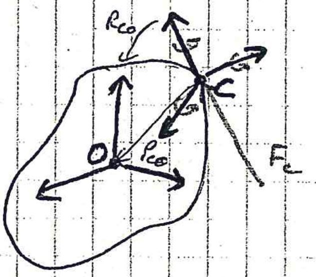
1 Wrenches and Grasps

A wrench is a generalised force acting at a point with linear (force) and angular (moment) components:

$$F = \begin{pmatrix} f \\ \tau \end{pmatrix}, \quad f, \tau \in \mathbb{R}^3$$

Write $F_c = \begin{pmatrix} f_c \\ \tau_c \end{pmatrix}$ if F is with respect to O_c.

Geometrical Motivation



O : Object
C : Contact

1. force only: $F_c = \begin{pmatrix} f_c \\ 0 \end{pmatrix}$

$g_{co} = (p_{co}, R_{co})$

Obvious: $F_o = \begin{pmatrix} f_o \\ \tau_o \end{pmatrix} = \begin{pmatrix} R_{co}^T f_c \\ R_{co}^T (-p_{co} \times f_c) \end{pmatrix}$

2. add τ_c: $F_c = \begin{pmatrix} 0 \\ \tau_c \end{pmatrix}$

$\hookrightarrow F_o = \begin{pmatrix} 0 \\ R_{co}^T \tau_c \end{pmatrix}$

3. general $F_c = \begin{pmatrix} f_c \\ \tau_c \end{pmatrix} = \begin{pmatrix} f_c \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \tau_c \end{pmatrix}$

$$\hookrightarrow F_o = \begin{pmatrix} R_{co}^T f_c \\ -R_{co}^T (p \times f_c) + R_{co}^T \tau_c \end{pmatrix}$$

$$= \begin{pmatrix} R_{co}^T & \\ -R_{co}^T \hat{p} & R_{co}^T \end{pmatrix} \begin{pmatrix} f_c \\ \tau_c \end{pmatrix}$$

$$= Ad_{g_{co}}^T \, F_c$$

We get: For g_{co} specifying the configuration of a contact C_i with respect to a body frame O the transpose of the adjoint Ad_g^T transforms wrenches F_c into equivalent wrenches F_o acting at the body frame:

$$F_o = Ad_{g_{co}}^T \; F_c$$

$$\left(\text{before: } \; V^o = Ad_{g_{oc}} V^c \right)$$

$$\left(\text{Note: } \; Ad_{g^{-1}} = Ad_g^{-1} \right)$$

Proof (of $F_o = Ad_{g_{co}}^T \, F_c$) by the principle of virtual work: (O_B : base frame)

infinitesimal work $\delta_W = V_{Bc}^b \cdot F_c$

V_{Bc}^b ist the object velocity with respect to the object in a base frame B_1

F_c the contact wrench

$\Rightarrow$ net work $W = \int_{t_1}^{t_2} V_{Bc}^b \, F_c \, dt$

and if replace F_c by F_o and V_{Bc}^b by V_{Bo}^b, the work remains the same.

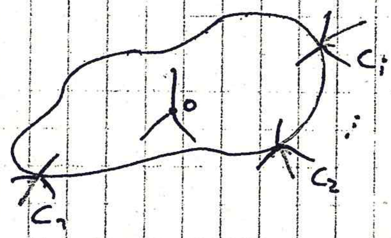
$$V_{Bo}^b \, F_o = V_{Bc}^b \, F_c = \left(Ad_{g_{co}} \, V_{Bo}^b \right)^T F_c$$

$$= V_{Bo}^{bT} \, Ad_{g_{co}}^T \, F_c$$

$$\Rightarrow F_o = Ad_{g_{co}}^T \, F_c$$

Typical problem:

Find the net wrench for a multifingered grasp



Compute $F_o = \sum \left(Ad_{g_{oc_i}^{-1}} \right)^T F_{c_i}$

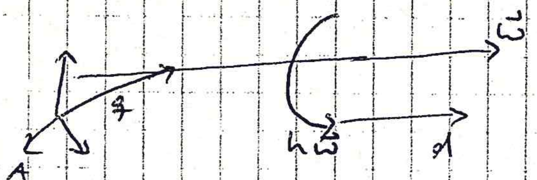
$$= \sum Ad^T_{g_{c_i o}} F_{c_i}$$

Screw coordinates for wrenches

Wrenches may act along screws:

Given $(l = q + \lambda \vec{\omega},\ h,\ M,\ \|\vec{\omega}\| = 1)$, then construct a wrench by application of a force of magnitude M along l and a torque hM about l:

$$F = \begin{pmatrix} \vec{\omega} \\ -\vec{\omega} \times q + h\omega \end{pmatrix} \cdot M \quad , \quad h < \infty \qquad (*).1$$



$$h = \frac{d}{\theta}$$

$$F = M \cdot \begin{pmatrix} 0 \\ \omega \end{pmatrix} \quad , \quad h = \infty \qquad (*).2$$

F_A is called the wrench along the screw specified w.r.t. to A because $\vec{q}, \vec{\omega}$ are specified in A.

To find the screw for a given wrench, solve (*) for M, h, $\vec{\omega}$, q given $F = \begin{bmatrix} f \\ \tau \end{bmatrix}$.

1) $(f = 0)$ set $M = \|\tau\|$

$$\vec{\omega} = \frac{\tau}{\|\tau\|}$$

$$h = \infty$$

2) $(f \neq 0)$ set $M = \|f\|$

$$\vec{\omega} = \frac{f}{\|f\|}$$

and solve $M(-\vec{\omega} \times q + h\vec{\omega}) = \tau$ for h, q:

eg: $h = \frac{f^T \tau}{\|f\|^2}$, $q = \frac{f \times \vec{\omega}}{\|f\|^2}$

1.2. Reciprocal screws

<u>Def:</u> Two screws s_1, s_2 are reciprocal if the twist V about s_1 and the wrench F about s_2 are reciprocal, i.e.

$$V \cdot F = 0$$

<u>Lemma:</u> s_1, s_2 are reciprocal if

screw product

$$s_1 \odot s_2 = M_1 M_2 ((h_1 + h_2) \cos\alpha - d \sin\alpha) = 0$$

where α is the angle between the screws' axis and d is the smallest distance:

$$\alpha = \operatorname{atan2}((\omega_1 \times \omega_2) \cdot n, \ \omega_1 \cdot \omega_2)$$

and n ist the direction along the smallest distance.

$\odot$ is called reciprocal product.

Interpretation:

A number of screws $s_1, .., s_N$ form a subspace of R^6 if linearly.

$\Rightarrow$ The set of reciprocal screws form a „reciprocal" subspace.

We have $\dim(\text{span}(s_1 \dots s_N))$
$+ \dim(\text{span}(\text{reciprocal system})$
$= 6$

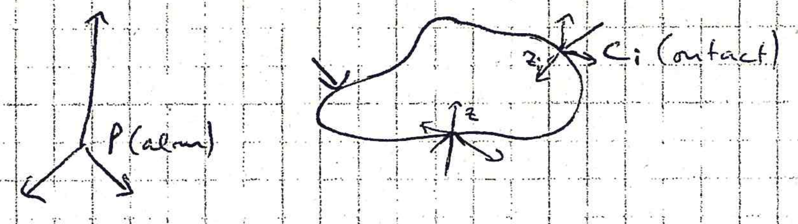
$\Rightarrow$ Interpreting screws as wrenches, the reciprocal system yields motions (via twists) which can not be resisted.

$\Rightarrow$ Interpretation of screws as twists yields wrenches in the reciprocal system, whic cause no motion.

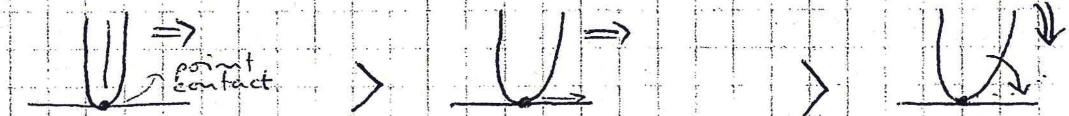
Zettel 5

1.3. Contacts and friction models

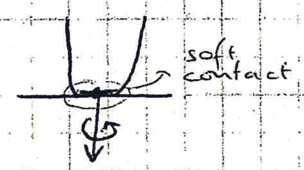
We consider (point) contact at fixed locations, i.e. a situation described by the coordinate systems of Fig. and g_{PO}, g_{OC_i}. By convention, the z-axis of the C_i-systems is pointing inwards along the surface normal.



Different contact models have to consider different friction types and properties of the finger:



soft finger contacts:



torsional friction $\Rightarrow$ allows also a torque around the surface norm

We consider three cases characterised by different wrench basis B and friction cones FC:

1) Frictionless point contact

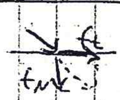
The applied wrench in C_i coordinates is:

$$F_{C_i} = \begin{bmatrix} 0 \\ 0 \\ f_i \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \cdot f_i \ , \ f_i \geq 0$$

wrench basis B_i

$FC_{c_i} = \{ f_{c_i} \in \mathbb{R}^+ \}$

2) Point contact with Coulomb friction

Write f as $f = f_N + f_t$: 

Coulomb's law:

slipping starts if $|f_t| > \mu f_n$

where μ is the friction coefficient of the materials.

Geometrically the set of forces which can be applied, are in a cone with angle $\alpha = \tan^{-1} \mu$.

The wrench applied can be written as

$$F_{c_i} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ & 0_{3\times 3} & \end{bmatrix}}_{\substack{\text{wrench} \\ \text{basis } B_i}} \cdot \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} = B_i \, f_{c_i}$$

$$FC_{c_i} = \left\{ f \in \mathbb{R}^3 \;\middle|\; f_3 > 0, \sqrt{f_1^2 + f_2^2} \leq \mu f_3 \right\}$$

3) Soft finger contact

We assume that additionally a torque around the surface normal can be applied:

$$F_{c_i} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot f_{c_i} \in FC_{c_i}$$

$$FC_{c_i} = \left\{ f \in \mathbb{R}^4 \;\middle|\; f_3 > 0, \sqrt{f_1^2 + f_2^2} \leq \mu f_3, \; |f_4| \leq \gamma f_3 \right\}$$

where γ is the torsional friction coefficient.

The general contact model is

$$B_{c_i} \in \mathbb{R}^{p \times m_i}$$

where p ist the dimension of the wrench space ($p = 6$ in 3D, $p = 3$ in 2D) and m_i is the number of independent forces, that can be applied, together with the friction cone FC_{c_i}.

For the friction cone it holds

$$f_1, f_2 \in FC_{c_i} \Rightarrow \alpha f_1 + \beta f_2 \in FC_{c_i} \text{ for } \alpha, \beta > 0$$

1.4. The grasp map

Consider a grasp defined by the configuration g_{po} and a number k of contact points g_{oc_i} and determine the net effect of the applied wrenches:

$$F_o = \sum Ad^T_{g_{oc_i}^{-1}} F_{c_i}$$

where $F_{c_i} = B_{c_i} f_{c_i}$, $f_{c_i} \in FC_{c_i}$

Then

$$F_o = \sum \begin{bmatrix} R_{oc_i} & 0 \\ \hat{p}_{oc_i} R_{oc_i} & R_{oc_i} \end{bmatrix} B_{c_i} f_{c_i}$$

Define $G_{c_i} = \Big[\quad '' \quad \Big] B_{c_i}$

to obtain

$$F_o = G_{c_1} f_{c_1} + \dots + G_{c_k} f_k = \vec{G}^T \vec{f}$$

<u>Def:</u> $G = \left[Ad^T_{g_{oc_1}^{-1}} B_{c_1}, \dots, Ad^T_{g_{oc_k}^{-1}} B_{c_k} \right] : \mathbb{R}^m \rightarrow \mathbb{R}^p$,

$m = m_1 + \dots + m_k$

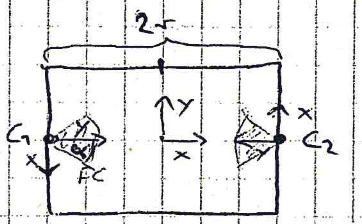
is called grasp map.

We write $F_o = G f_c$, where $f_c = (f_{c_1} \dots f_{c_k})^T \in \mathbb{R}^m$

Write $FC = FC_{c_1} \times FC_{c_2} \times \dots \times FC_{c_k} \subseteq \mathbb{R}^m$.

<u>Def:</u> A grasp is a pair G, FC.

<u>Example:</u> Planar grasping of a box by means of 2 contacts with friction:



$$F_o = \begin{bmatrix} f_o \\ \tau_o \end{bmatrix} = \sum_{i=1,2} \begin{bmatrix} R_{oc_i} & \begin{matrix} 0 \\ 0 \end{matrix} \\ [-p_y, p_x] R_{oc_i} & 1 \end{bmatrix} \begin{bmatrix} f_i \\ \tau_i \end{bmatrix}$$

$R_{oc_i} \in SE(2)$, $f_{c_i} \in \mathbb{R}^2$, $\tau_{c_i} \in \mathbb{R}$

$$R_{oc_1} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad R_{oc_2} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$p_{oc_1} = \begin{bmatrix} -r \\ 0 \end{bmatrix}, \quad p_{oc_2} = \begin{bmatrix} r \\ 0 \end{bmatrix}$$

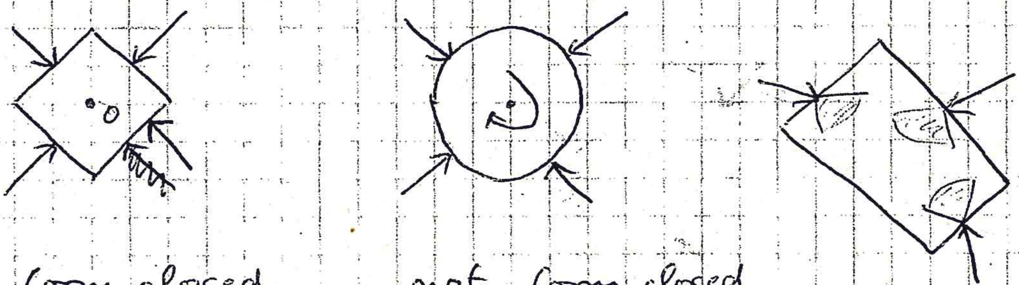
$$B_{c_i} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$G = \begin{bmatrix} 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ r & 0 & r & 0 \end{bmatrix}$$

2 Grasp Properties

2.1. Form Closure

Def.: A grasp is form closed if it can resist any external disturbance wrench, irrespective of the magnitude of the frictional contact forces.



form closed

not form closed because rotation

not form closed because here we rely on friction!

There are construction conditions to evaluate form closure: Compute the grasp matrix for frictionless contacts:

$$F_{O_i} = \begin{bmatrix} R_{OC_i} & 0 \\ \hat{p}_{OC_i} R_{OC_i} & R_{OC_i} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} f_i$$

$$G = \left[\begin{bmatrix} n_1 \\ p_1 \times n_1 \end{bmatrix}, \ldots, \begin{bmatrix} n_N \\ p_N \times n_N \end{bmatrix} \right] \in \mathbb{R}^{6 \times N}$$

To resist arbitrary wrenches in $\mathbb{R}^6$ we have to require that $G(FC)$ span the whole $\mathbb{R}^6$.

Lemma: A grasp is form closed

$\Leftrightarrow$ the columns of G positively span $\mathbb{R}^6$, i.e.

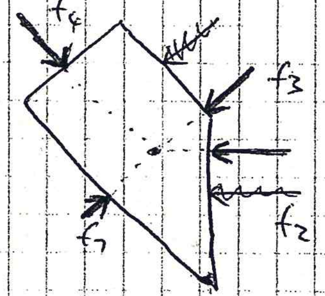
$\forall x \in \mathbb{R}^6$ there exist $\alpha_i > 0$ such that

$\sum \alpha_i \, col_i(G) = x$

$\Leftrightarrow$ the origin is an inner element of the convex hull of $col(G)$, i.e.

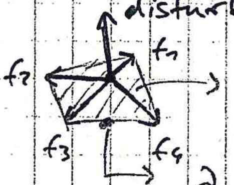
$\exists \; \lambda_i > 0 \;, \; \sum \lambda = 1 : \sum \lambda_i \, col(G) = 0$

$\wedge \;\; 0 \notin \partial \, co(col_i(G))$



$\Leftarrow$ no moments

$\Rightarrow$ $col_i(G_i) = \begin{bmatrix} n_i \\ 0 \\ 0 \\ 0 \end{bmatrix}$ im wrench space!



„konvexe Hülle"

$co(col_i(G))$

$\partial \, co(col_i(G))$

$\Rightarrow$ it is always possible to find a positive combination of $col_i(G)$ which points in the negative direction of the disturbance force.

$\Rightarrow$ torques can not be resisted in this example.

$\Rightarrow$ if the wrench space has dim N, then we need at least $N+1$ points contacts for form closure.

Remark: There are lots of algorithm to evaluate the convexity condition.

2.2. Force closure and internal forces

Def.: A grasp is force closed if for any external disturbance wrench F_e there exists a $f \in FC$ such that $Gf = -F_e$

We have: form closure $\Rightarrow$ force closure

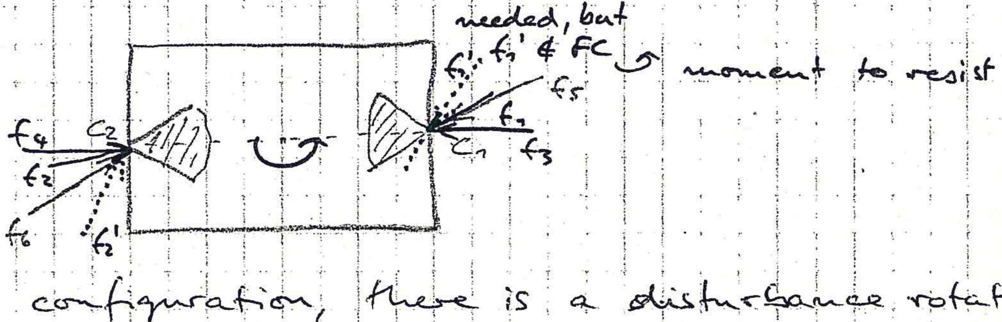
form closure $\not\Leftarrow$ force closure

Force closure grasps are characterized by internal forces f_N which cause no net effect on the object, i.e.:

$$G f_N = 0 \iff f_N \text{ in the nullspace } N(G) \text{ of } G.$$

On the otherhand, if $f_N \in N(G)$, then it is calle an internal force; if $f_N \in$ {interior $(N(G) \cap FC)$, then it is called strictly internal force.

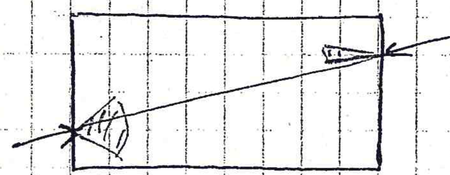
An internal force is a set of forces represented with respect to the contact frames and the wrench basis at each contact.



In this configuration, there is a disturbance rotation which can not be resisted with fixed force magnitude.

But using internal forces f_3, f_4, the resulting f_5, f_6 are inside the friction cones.

If the „potentionally" counteracting forces are not inside the friction cone FC, then there is no force closure:



Lemma: A grasp is force closed if and only if G is surjective and $\exists\, f_N \in \operatorname{int}(N(G) \cap FC)$

„$\Leftarrow$" : Murray p. 224

„$\Rightarrow$" : Assume we have force closure.

Then choose for F_e

$f_1: \quad G f_1 = F_e \neq 0 \quad , f_1 \in FC$

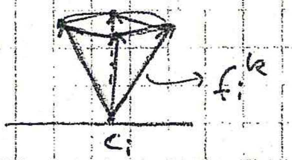
$\Rightarrow f_2: \quad G f_2 = -F_e \neq 0 \quad , f_2 \in FC$

$\Rightarrow f_N = f_1 + f_2$ is internal

How to check force closure?

Problem: Forming the convex hull for 3D cones is not directly possible.

⇒ approximate the 3D-cones by a finite number of k boundary vectors:



⇒ Then treat all f_i^k as point contacts and form the convex hull for $F_{OC_i}^k = \left[Ad^T_{g^{-1}_{OC_{i}k}} f_i^k \right] = \begin{bmatrix} f_i^k \\ p_i \times f_i^k \end{bmatrix}$

The grasp is force closure if $0 \in co(\{F_{OC_i}^k\})$

2.3. Quality measures

Most of such measures are based on the convex hull approximation for force closure. For comparison consider normalized wrenches (forces). Two approaches:

(1) $\| f_{c_i} \| = 1$

(2) $\| \Sigma f_{c_i} \| = 1$ (easier, because convex hull can be used directly)

↳ define $W = \{$ set of all wrenches possible to be applied under $\| \Sigma f_{c_i} \| = 1 \}$

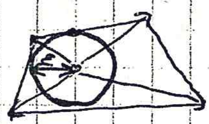
$$= co\left(\left\{\begin{bmatrix} f_i^k \\ p_i \times f_i^k \end{bmatrix}\right\}\right)$$

as for testing force closure.

Remark: It was proposed to use a scaling factor $\begin{bmatrix} f_i^k \\ \lambda(p_i \times f_i^k) \end{bmatrix}$, eg $\lambda = \frac{1}{r}$ proportional to the radius r of the object to achieve scale invariance.

measures:

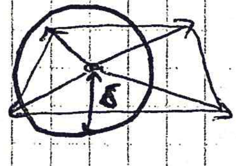
1) largest ε-ball fully inside the wrenchpolytope



The direction where the radius touches the border of the polytope gives also the worst disturbance.

Problem: Not independant to the torque origin
→ use center of mass.

2) use the radius of the average δ-ball



3) define V^F, the projection of the convex hull to the 3 force dimensions only or to

4) V^M for torque dimension + looking at ε or δ.

Remarks:

- yield criteria to optimize choice of grasping points: maximize ε w.r. to $f_i \in FC_i$
  ⇒ there are efficient LMI (Linear Matrix inequality) algorithms to do this [Frinkle 99]

But: Geometrical information about the object and possible (task-dependent) disturbances is lost!

2.4. Pertubation closure

Idea: compute for given object and given set of disturbances all changes of necessary grasp forces:

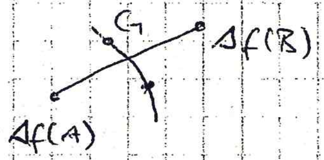
1) derive a formula, how fingertip forces f_i change for a given pertubation F, r:

$$\Delta f_i = \mathcal{F}(F, r, G)$$ (possible → Har/Wu 1999)

2) (2D)

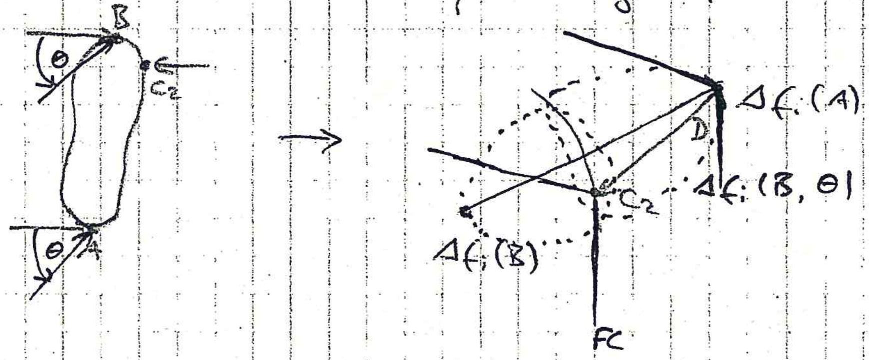
a) Move F at fixed orientation along the object conture, then the „worst case" for Δf_i occures at A and B, where F is tangential.

⇒ draw $\Delta f(F, r)$



(s. Skript ?)

b) now change orientation of F at A and B + draw the corresponding $\Delta f_i(A, \theta)$ curve



c) now geometrically find needed counterforce. Draw lines parallel to the friction cone at C_i and tangential to the graphs of $\Delta f_i(A, \theta$

$\Delta f_i(B,\theta)$. The vector (D, C_i) form the intersection to the contact point yields the maximal needed force to counteract the given F_A along the object conture for all possible θ.

$\Rightarrow$ Quality measure

$$m = \frac{M}{\max(\|\Delta f_i\|)} \quad ; \quad M = \|F\| \text{ for given set of } \theta \text{ and a conture.}$$

It can be shown:

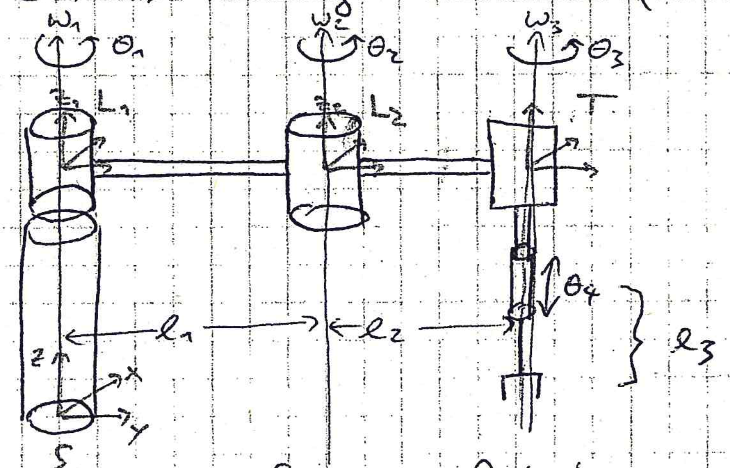
$$0 < m < n \cdot \frac{\mu}{(1+\mu)^2}$$ holds, where μ is a uniform friction coefficient.

4 Kinematics and Manipulation

4.1. Motivation: Manipulation needs Kinematics

Consider manipulators with revolute joints (angles $\theta_i \in Q_i$, $Q_i = [0, 2\pi]$) and also prismatic joints (displacements $\theta_i \in Q_i = \mathbb{R}$)

Define for k joints the joint space Q as $Q = Q_1 \times Q_2 \times \dots \times Q_k$ and chose a base frame S, a tool frame T and intermediate frames with z-axis along the axis of motion for each joint



Scara - Robot

Then

$$g_{st} : Q \rightarrow SE(3)$$

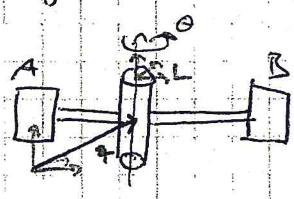
is the forward kinematics mapping. We can generate g_{st} in different ways:

1) conventionally by using relative configuratio[n]

$$g_{st}(\theta_1 \dots \theta_k) = g_{sl_1}(\theta_1) \cdot g_{l_1 l_2}(\theta_2) \cdots g_{kT}(\theta_k)$$

2) directly use the twists associated with screw motions in the joints with twist coordinates ξ

$$g_{ab}(\theta) = \exp(\hat{\xi}\theta) \cdot g_{ab}(0)$$



$$, \quad \xi = \begin{bmatrix} -\omega \times q \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$\Delta f_i(B, \Theta)$. The vector (D, C_i) form the intersection to the contact point yields the maximal needed force to counteract the given F_{Θ} along the object conture for all possible Θ.

$\Rightarrow$ Quality measure

$$m = \frac{M}{\max(\|\Delta f_i\|)} \quad ; \quad M = \|F\| \text{ for given set of } \Theta \text{ and a conture.}$$

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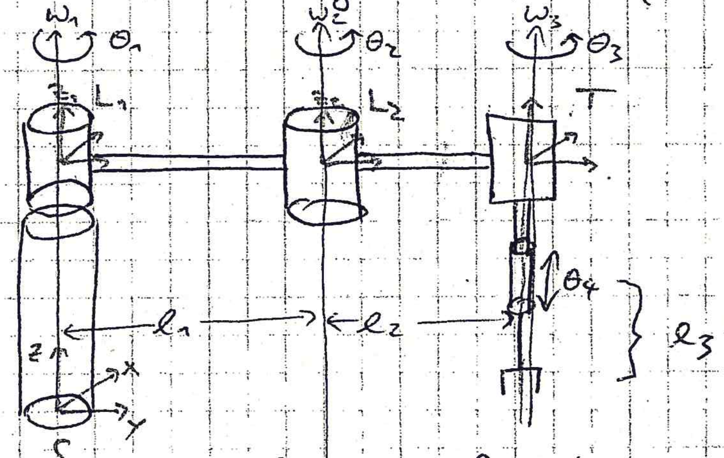

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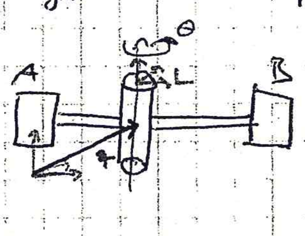
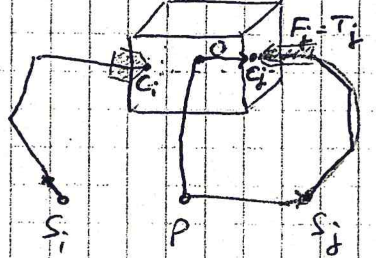


$$, \quad \xi = \begin{bmatrix} -\omega \times q \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

4.3. Finger Kinematics and grasp constraints

· Assume fixed contact points, then the kinematics yield constraints:
the velocities of the fingertips (tool frame) and the velocities of the respective contact frames have to be equal.



Model the hand as a collection of fingers w.r. to a common palm P, then F_i moves with the i-th finger, whereas C_i moves with the object. The constraint requires that the relative velocity between F_i and C_i must be zero. in directions defined by the contact model B_{C_i}:

$$B_{C_i}^T \, V_{F_i C_i}^b = 0 \qquad (*)$$

Now expand $V_{F_i C_i}^b$ in quantities we know (body velocities, G, J_H) using P.

Remember: $(a \to b \to c)$ $F_i \to P \to C_i$

$$V_{ac}^b = Ad_{g_{bc}}^{-1} V_{ab}^b + V_{bc}^b \qquad (1)$$

$$V_{ab}^b = -Ad_{g_{ab}} V_{ba}^b \qquad (2)$$

Robotic 15.07.03 / Bsg

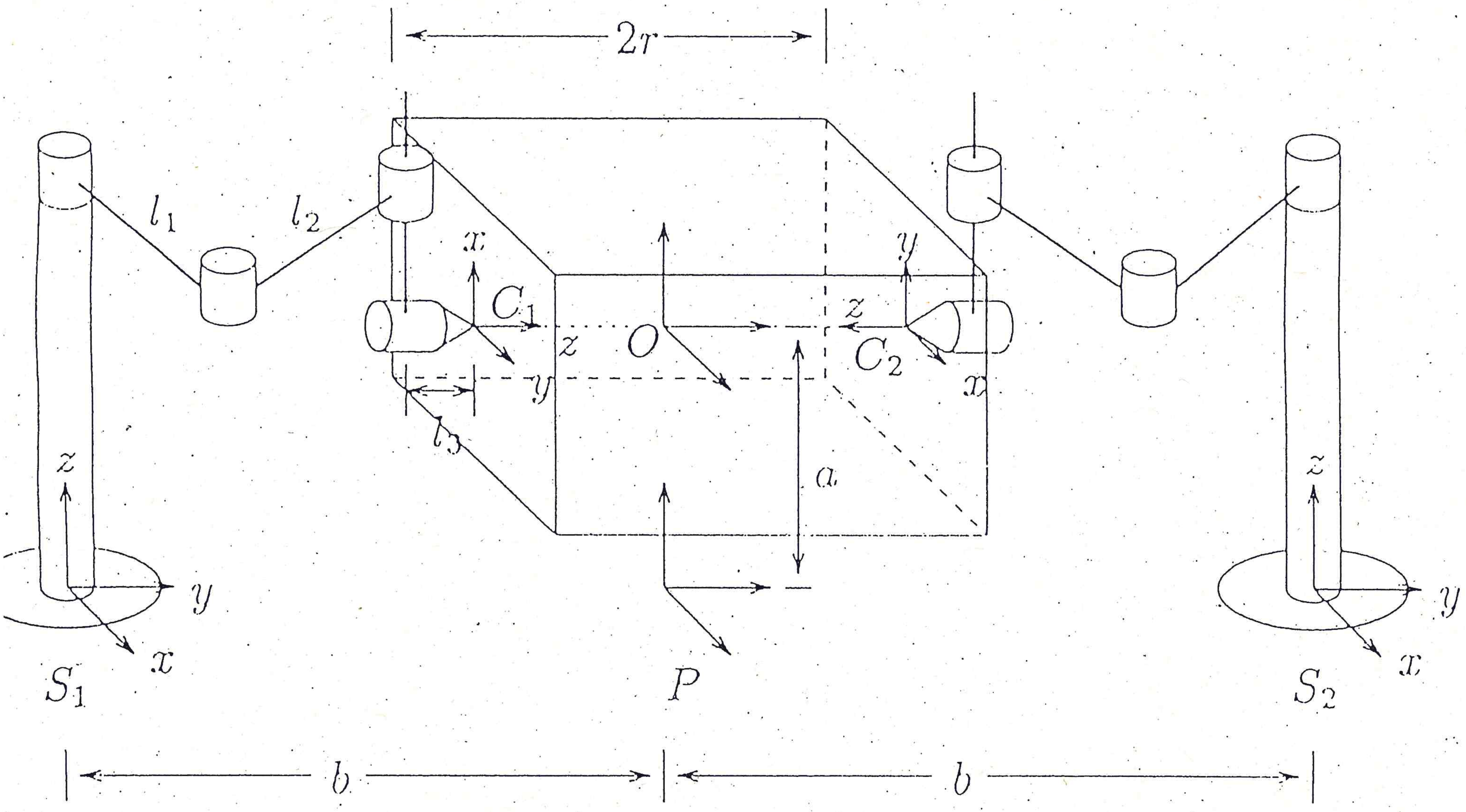
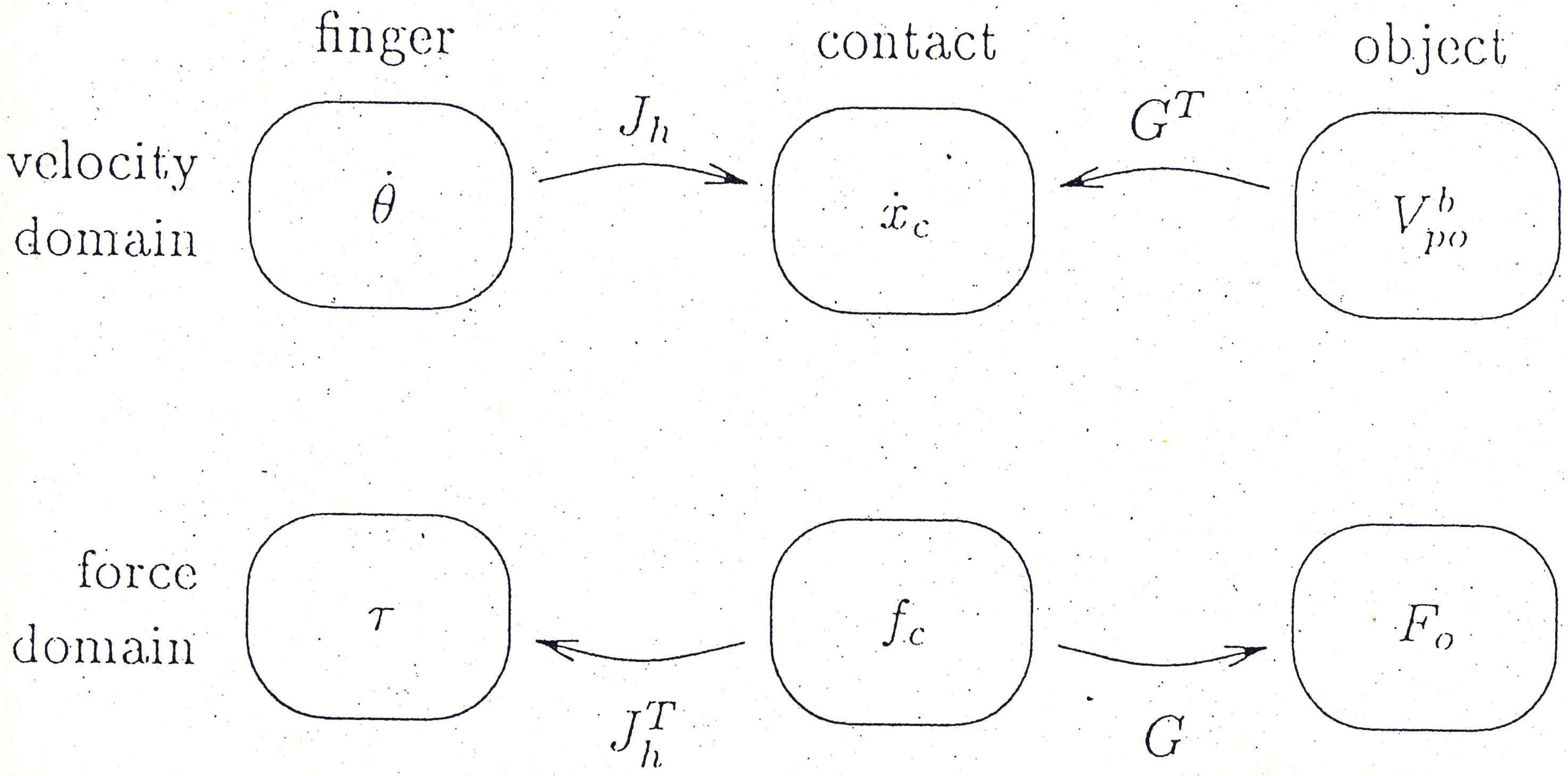


Figure 5.17: Two-fingered grasp using SCARA robots.



ure 5.15: Diagram of relationships for a multifingered grasp. The tact force must satisfy $f_c \in FC$ for these relationships to hold.

(Skip the „b"-superscript for body-velocity)

$$V_{F_iC_i} \overset{(1)}{=} \underline{Ad_{g_{PC_i}}^{-1} \, V_{F_iP}} + \underline{V_{PC_i}}$$

$$\overset{(2)}{=} \underline{- Ad_{g_{PC_i}}^{-1} \, Ad_{g_{PF_i}} \, V_{PF_i}} + \underline{V_{PC_i}}$$

Also: $\quad V_{PC_i} \overset{(1)}{=} Ad_{g_{OC_i}}^{-1} \, V_{PO}$

Substitute in the constraint:

$$B_{C_i}^T \left(Ad_{g_{PC_i}}^{-1} \, Ad_{g_{PF_i}} \right) V_{PF_i}^b$$

$$= \underline{\underbrace{B_{C_i}^T \, Ad_{g_{OC_i}}^{-1}}_{[G^T]_i} \, V_{PO}^b}$$

$$= \underline{B_{C_i}^T \underbrace{\left(Ad_{g_{PC_i}}^{-1} \, Ad_{PF_i} \, Ad_{g_{S_iF_i}}^{-1} \right)}_{Ad_{g_{S_iC_i}}^{-1}} V_{S_iF_i}^S}$$

$$= B_{C_i}^T \, Ad_{g_{S_iC_i}}^{-1} \, J_{S_iF_i}^S \, \dot{\vec{\theta}}^i$$

$$\Rightarrow \quad [G^T]_i \, V_{PO}^b = B_{C_i}^T \, Ad_{g_{S_iC_i}}^{-1} \, J_{S_iF_i}^S \, \dot{\vec{\theta}}^i$$

where $\dot{\vec{\theta}}^i$: Vector of joint velocity for i-th finger

<u>Definition</u> Define the hand-Jacobian for k contacts as

→ s. Übungszettel 9 → maple-File

$$J_h(\vec{\theta}, X_o) = \begin{bmatrix} B_{C_1}^T \, Ad_{g_{S_1C_1}}^{-1} \, J_{S_1F_1}^S (\vec{\theta}^1) & & 0 \\ & \ddots & \\ 0 & & B_{C_k}^T \, Ad_{g_{S_kC_k}}^{-1} \, J_{S_kF_k}^S (\vec{\theta}^k) \end{bmatrix}$$

$J_h(\theta, X_o) \; : \; \mathbb{R}^n \rightarrow \mathbb{R}^m$

$n = n_1 + \dots + n_k$, n_k is the number of joints of the i-th finger, $\vec{\theta}^i \in \mathbb{R}^{n_i}$ and X_o is the default configuration g_{PO}

and the fingers $g_{s_i f_i}$. Now write the grasp constraint as

$$\boxed{\underline{J_h(\vec{\theta}, x_o)\,\dot{\vec{\theta}}} = \underline{G^T V_{PO}^b}} \qquad (**)$$

4.4. Manipulability

Evaluate the possibility of the finger moving the object by properties of J_h:

Def.: A multifingered grasp is manipulable at a configuration $(\vec{\theta}, x_o)$, if for any object motion V_{PO}^b there exists a $\dot{\vec{\theta}} = [\dot{\vec{\theta}}^{1T}, \dot{\vec{\theta}}^{2T}, \ldots]^T$ satisfying $(**)$.

It holds:

range = Rang?

grasp manipulable $\Leftrightarrow$ $R(G^T) \subseteq R(J_h(\theta, x_o))$

Remarks:

- for manipulation, J_h must be full row rank
- in case $n > m$ (redundancy) there exist $\dot{\vec{\theta}} \in \overset{\text{nullspace}}{N}(J_h)$ called internal motions, which have no effect on the object velocity.
- redundancy can (also) appear by combination of non-redundant fingers.
- then use dynamics!

4.5. Structural Forces

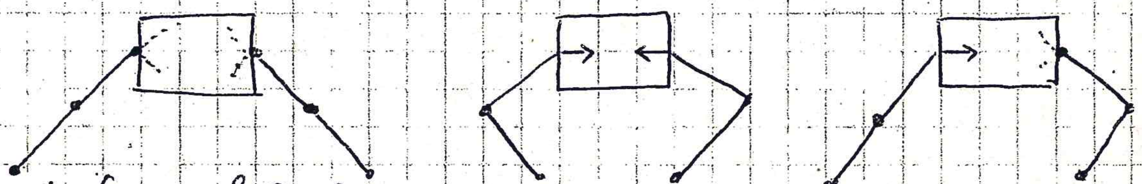
The hand-Jacobian also relates (object) forces to joint torques (by the principle of virtual work)

$$\tau = J_h^T f_{c_{\varepsilon}} \quad , \quad f_c \in FC$$

In case the grasp is not manipulable, J_h^T has a non trivial null-space, i.e. there exist $f_c \in N(J_h^T)$ with $0 = J_h^T f_c$.

Such forces are called structurally dependend forces, since they generate forces in the physical mechanism (which may be evaluated using the elastic properties of the materials and mechanisms involved)

<u>Remark:</u> Manipulability and force closure are independent!



+ force closure
− manipulability

− force closure
+ manip.

− force cl.
− manip.

Example (Bild sich Kopie!) → s. Übung + Maple-File

grasp map:

$$G = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ -r & 0 & 0 & 0 & 0 & r & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & r & 0 & 0 & -r & 0 & 0 & 0 \end{bmatrix} \quad \rightarrow \text{rank?}$$

$$B_{c_i} = \begin{bmatrix} I & \begin{matrix}0\\0\\0\\0\\0\\1\end{matrix} \\ 0 & \end{bmatrix}, \quad l_3 \overset{!}{=} 0$$

$$J^s_{s_i f_i} = \begin{bmatrix} 0 & l_1 c_1 & l_1 c_1 + l_2 c_{12} & 0 \\ 0 & l_1 s_1 & l_1 s_1 + l_2 s_{12} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

$$g_{po}: \quad R_{op} = I, \quad p_{op} = \begin{bmatrix} 0 \\ 0 \\ a \end{bmatrix}$$

$$Ad^{-1}_{g_{s_i c_i}} = Ad^{-1}_{g_{oc_i}} \, Ad^{-1}_{g_{po}} \, Ad_{g_{s_i p}}$$

<u>results:</u>

$$Ad^{-1}_{g_{s_1 c_1}} = \begin{pmatrix} R^T_{s_1 c_1} & \begin{pmatrix} b-r & 0 & 0 \\ 0 & a & r-b \\ -a & 0 & 0 \end{pmatrix} \\ 0 & R^T_{s_1 c_1} \end{pmatrix}, \quad R_{s_1 c_1} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

$$Ad^{-1}_{g_{s_2 c_2}} = \begin{bmatrix} R^T_{s_2 c_2} & \begin{pmatrix} 0 & a & b-r \\ r-b & 0 & 0 \\ a & 0 & 0 \end{pmatrix} \\ 0 & R^T_{s_2 c_2} \end{bmatrix}, \quad R_{s_2 c_2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$J_h = \begin{bmatrix} J_{11} & \\ & J_{22} \end{bmatrix}, \quad J_{11} = \begin{bmatrix} & & 0 & \\ 0 & 0 & -b+r+l_1 c_1 + l_2 q_2 & 1 \\ -b+r & -b+r+l_1 c_1 & l_1 s_1 + l_2 s_{12} & 0 \\ 0 & l_1 s_1 & & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \quad G^T \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ -1 \end{bmatrix} \notin R(J_h) = \text{The grasp is } \underline{\text{not}} \text{ manipulable.}$$

5 Simulation of Grasping

Goals: • test algorithms offline without involving the costly hardware for:
- choice of grasp points
- force optimisation
- quality measures
- geometric feasibility
- lerning
- ...

Issues for simulation:

- geometry: objects, hand
- render (takes most computing time!)
- kinematics (generate joints)
- sensors (forces, position, velocity, cameras, ...)
- friction + collisions (→ efficiency important!)
- material properties

up to here is everything static and „many" people have this „in house", most prominent GRASPIT!
www.cs.columbia.edu/~amiller/graspit

- simulate the dynamics
  - rigid body dynamics
  - friction (static vs. sliding vs. rolling)
  - actuators (motors etc.)

Applications: Games, Fun, Science

Here look at simulating the dynamics only!

General approach:

Assume that a velocity constraint is given:

$$\dot{C} = 0 = \frac{\partial C}{\partial q}\,\dot{q} \qquad \text{where}$$

C is a funtion of state variables q. Then $\frac{\partial C}{\partial q} = J$ is called the constraint Jacobian.

Then:

$$\ddot{C} = \dot{J}\dot{q} + J\ddot{q} \;,\quad \dot{J} = \frac{\partial \dot{C}}{\partial q}$$

and

$$\ddot{q} = \underbrace{W}_{\substack{\text{material properties} \\ \text{mass} \\ \text{inertia tensor}}}\left[\begin{pmatrix} F \\ M \end{pmatrix} + \underbrace{\begin{pmatrix} \hat{F} \\ \hat{M} \end{pmatrix}}_{\text{constraint force / moment}}\right]$$

$$\ddot{C} \stackrel{!}{=} 0$$

$$\Rightarrow J W \begin{pmatrix} \hat{F} \\ \hat{M} \end{pmatrix} = -\dot{J}\dot{q} - J W \begin{pmatrix} F \\ M \end{pmatrix}$$

$$\left(\frac{\hat{f}^x}{m} = -\dot{x}\dot{x} - \frac{f^x}{m} \right)$$

As before by the virtual work principle:

$$\begin{pmatrix} \hat{F} \\ \hat{M} \end{pmatrix} \dot{q} = 0 \;,\quad \forall \dot{q} : \underbrace{J\dot{q}}_{\dot{C}=0} = 0$$

$$\Rightarrow \begin{pmatrix} \hat{F} \\ \hat{M} \end{pmatrix} = J^T \vec{\lambda} \;,\quad \vec{\lambda} \in \mathbb{R}^{\dim C}$$

$J^T\lambda$ is normal to the hypersurface $C=0$ because the rows of the Jacobian are the gradients of the C_i-functions, and $J^T\lambda$ forms a linear combination of gradient direction (normals).

$$J v = J\dot{q} = 0 \quad \text{are the legal velocities}$$

and the orthogonal complement $J^T\lambda$ can counteract deviations from these.

. Finally solve for λ:

$$J W J^T \lambda = -\dot{J}\dot{q} - J W \binom{F}{\mu} \qquad (*)$$

(linear systems to be solved for λ in each timestep numerically)

Then compute acceleration etc.

<u>Example:</u>

Contact constraints leads to velocity condition:

O_1, O_2 are centers of masses of objects.

$\rho_{O_1C_1}$, $\rho_{O_2C_2}$ are the vectors for the contact points.

$$C(\dot{q}) = \vec{n}^T v_1 + \vec{n}^T \omega_1 \times \rho_{O_1C_1} - \vec{n}^T v_2 + \vec{n}^T \omega_2 \times \rho_{O_2C_2} = 0 \quad , \vec{n}: \text{normal}$$

$$\Rightarrow J \underbrace{\begin{pmatrix} v_1 \\ -v_2 \\ \rho_1 \\ \rho_2 \end{pmatrix}}_{\dot{q}} = 0 \quad , \quad J = \begin{pmatrix} \vec{n}^T \\ -\vec{n}^T \\ \vec{n}^T \hat{\omega}_1 \\ \vec{n}^T \hat{\omega}_2 \end{pmatrix} = \frac{\partial C}{\partial q}$$

Now proceed with $(*)$

Simulation issues

Typical problem: instability due to numerical inaccuracy and approximations (mass, inertia, constraints never fulfilled..)

$\Rightarrow$ constraints are never exactly maintained.

Approaches to cope with this:

- add in the velocity constraint error correction terms:

  $$n^T(v_1 - v_2 + \omega_1 \times p_1 - \omega_2 \times p_2) = \frac{\varepsilon}{\Delta t}\big(\underbrace{v_1 - v_2 + (\omega_1 - \omega_2)}_{\text{error}}\big)$$

  $\varepsilon = 1$ would correct the error in one time step, but once more generates instability (wants to be exact in next time step $\rightarrow$ does not work)

  Recommanded is $0{,}1 < \varepsilon < 0{,}8$

- add terms to allow interpenetration:

  $$J \cdot \dot{q} = \frac{\varepsilon}{\Delta t}(\cdot) + \delta \begin{pmatrix} \hat{F} \\ \hat{\mu} \end{pmatrix} \quad \text{where}$$

  $\begin{pmatrix} \hat{F} \\ \hat{\mu} \end{pmatrix}$ is normal to the constraint surface.

  Proper settings of ε, δ allow an „elastic spring" behaviour with damping! (implemented in Vortex)

- compute $*$ only approximately by neglecting $\dot{J}\dot{q}$ terms. ~~and also~~

Collision detection

Geometric (hard) ~~obje~~ problem:

for n objects there are n^2 potential collisions,

complex objects have many potential collision point

Different approaches to simplify the problems:

- velocity constraints
- known analytic trajectories
- sparse environment
- specilised objects (box, balls, cylinders)
- convex objects
- approximation by outer boxes (in robotics to ensure safety regions, i.e. to avoid collisions)
- release real time constraint

A general (fast) solution can be obtained in steps:

(1) pairwise collisions of convex objects

(2) extension to multiple objects by efficient search of candidate pairs.

(3) use hierarchical models of combinations of convex objects.

Here consider (1) only.

<u>Idea:</u> use polytopes only and describe it by features (vertices, edges, faces) and track the pair of closest features.

<u>Ansatz:</u> Compute Voronoi-Regions $V(f)$ for each feature f, i.e. the set of points closest to that feature, they are defined by planes and always share planes with neighboring features.

Then, if a point p_A on object A is in the Voronoi-region of feature f_B on B <u>and</u> vise versa, then f_A and f_B are closest.

The Voronoi-regions and the feature data structure can be precomputed.

Track closest features by:

1) Precompute regions
2) Pick randomly two features
3) Check if $f_B \in V(f_A)$ and $f_A \in V(f_B)$
4) If not, pick better neighbors
5) go to 3.

This involves local tests only → <u>is fast</u>

⇒ available in free code: I-COLLIDE!

~~7.7. Information und Lernen~~

~~<u>Information</u> besitzt immer eine Unsicherheit, indem aus versch. Mögl. (Nachrichten) eine ausgewählt wird.~~

~~<u>Maß</u> für Information – gewünschte Eigenschaften:~~

- ~~sicheres Ereignis $\vec{x}_i$ (mit $P(\vec{x}_i) = 1$) enthält geringe Information~~
- ~~unwahrscheinliches Ereignis ($P(\vec{x}_i) \ll 1$) enthält große Information.~~